

# COMPARISON OF 3-DIMENSIONAL $\mathbb{Z}$ -GRADED AND $\mathbb{Z}_2$ -GRADED $L_\infty$ ALGEBRAS

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**ABSTRACT.** This article explores  $L_\infty$  structures – also known as ‘strongly homotopy Lie algebras’ – on 3-dimensional vector spaces with both  $\mathbb{Z}$ - and  $\mathbb{Z}_2$ -gradings. Since the  $\mathbb{Z}$ -graded  $L_\infty$  algebras are special cases of  $\mathbb{Z}_2$ -graded algebras in the induced  $\mathbb{Z}_2$ -grading, there are generally fewer  $\mathbb{Z}$ -graded  $L_\infty$  structures on a given space. On the other hand, only degree zero automorphisms, rather than just even automorphisms, are used to determine equivalence in a  $\mathbb{Z}$ -graded space. We therefore find nontrivial examples in which the map from the  $\mathbb{Z}$ -graded moduli space to the  $\mathbb{Z}_2$ -graded moduli space is bijective, injective but not surjective, or surjective but not injective. Additionally, we study how the codifferentials in the moduli spaces deform into other nonequivalent codifferentials, which gives each moduli space a sort of topology.

## 1. INTRODUCTION

Infinity algebras commonly arise in both  $\mathbb{Z}$ -gradings and  $\mathbb{Z}_2$ -gradings. For example, in the study of supersymmetry, a  $\mathbb{Z}_2$ -grading is natural. In topological or algebraic contexts, though, a  $\mathbb{Z}$ -grading is often preferable. In this article, we explore the relationship between  $L_\infty$  structures with these two different gradings. Because both the  $\mathbb{Z}$ -graded and  $\mathbb{Z}_2$ -graded  $L_\infty$  algebras on three-dimensional spaces have been thoroughly studied in [3, 1, 6, 5, 7], we will use  $L_\infty$  structures on these spaces to illustrate this relationship.

Every  $\mathbb{Z}$ -graded space has a natural  $\mathbb{Z}_2$ -grading, and since the signs which arise in the computations depend only on this  $\mathbb{Z}_2$ -grading, every  $\mathbb{Z}$ -graded infinity algebra induces a  $\mathbb{Z}_2$ -graded infinity algebra. Because of additional restrictions on the structures in the  $\mathbb{Z}$ -graded case, not

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every  $\mathbb{Z}_2$ -graded infinity algebra on a space will be a  $\mathbb{Z}$ -graded infinity algebra. From a naive point of view, it would seem that the  $\mathbb{Z}$ -graded infinity algebras are just a subset of the  $\mathbb{Z}_2$ -graded infinity algebras.

However, the moduli space of infinity algebras consists of equivalence classes of isomorphic structures, and the notion of isomorphism depends on whether the structure is considered as  $\mathbb{Z}$ -graded or  $\mathbb{Z}_2$ -graded. Since every isomorphism in the  $\mathbb{Z}$ -graded category is also an isomorphism in the  $\mathbb{Z}_2$ -graded category, there is a natural map from the moduli space of  $\mathbb{Z}$ -graded infinity algebras to the moduli space of  $\mathbb{Z}_2$ -graded infinity algebras. The purpose of this paper is to explore this map. We shall see that in some cases the map is bijective, but in others the map may fail to be injective or surjective.

In the  $\mathbb{Z}_2$ -graded category, there are only a few types of three-dimensional vector spaces. The dimension of a  $\mathbb{Z}_2$ -graded space is denoted in the form  $m|n$ , where  $m$  is the number of even basis elements and  $n$  is the number of odd basis elements. Thus, three-dimensional  $\mathbb{Z}_2$ -graded algebras are all of dimension  $3|0$ ,  $2|1$ ,  $1|2$  or  $0|3$ . There are an infinite number of possible three-dimensional  $\mathbb{Z}$ -graded spaces, which may be identified with a triple  $(k, m, n)$  of nondecreasing integers, representing the  $\mathbb{Z}$ -graded degrees of the three basis vectors. Each  $\mathbb{Z}$ -graded space also has a  $\mathbb{Z}_2$ -graded dimension, depending on which basis vectors are even and odd. Most  $\mathbb{Z}$ -graded spaces have only a finite number of  $L_\infty$ -algebra structures. However, a few special cases where the integers are consecutive (studied in [3]) or where two of the integers coincide (see [2]) have a richer moduli space. We will concentrate our study on these more interesting cases.

It can easily be shown that any nontrivial  $L_\infty$  structure on a vector space of type  $0|3$  must be a classical Lie algebra, and that there are no nontrivial  $L_\infty$  structures on spaces of type  $n|0$  (for arbitrary  $n$ ), so we do not have to consider spaces of type  $3|0$ . Thus, the interesting examples are of  $\mathbb{Z}$ -graded spaces of  $\mathbb{Z}_2$ -graded dimension  $1|2$  or  $2|1$ , which allows us to draw upon earlier work concerning  $\mathbb{Z}_2$ -graded spaces. In [5],  $L_\infty$  structures on  $\mathbb{Z}_2$ -graded spaces of dimension  $1|2$  were completely classified, and [1] classified  $L_\infty$  structures on  $\mathbb{Z}_2$ -graded spaces of type  $2|1$ .

Since the  $\mathbb{Z}_2$ -grading determines the signs in the computation of brackets, it is not surprising that  $\mathbb{Z}$ -graded spaces which are of dimension  $2|1$  in the induced  $\mathbb{Z}_2$ -grading have a completely different bracket structure than those whose induced  $\mathbb{Z}_2$ -grading is of dimension  $1|2$ . Accordingly, we will treat these two cases separately. It is interesting to note that since the signs in the bracket computations are entirely determined by the induced  $\mathbb{Z}_2$ -grading, the brackets of elements in a

$\mathbb{Z}$ -graded space can be calculated using exactly the same formulas that are used in the  $\mathbb{Z}_2$ -graded case.

The structure of this paper is as follows. Sections 2 through 4 cover preliminary notions about the moduli space of  $L_\infty$  algebras and the deformation theory of these algebras. In section 5, we consider the most interesting  $\mathbb{Z}$ -graded example of dimension  $1|2$ , the vector space of type  $(-1, 0, 1)$ , in considerable detail. In this particular example, we show that the  $\mathbb{Z}$ -graded moduli space is the same as the  $\mathbb{Z}_2$ -graded moduli space, down to the smallest detail that can be obtained from the deformation picture.

In section 6, we will consider  $\mathbb{Z}$ -graded spaces of dimension  $2|1$ , where there are two interesting examples. The moduli space of the  $\mathbb{Z}$ -graded vector space of type  $(0, 1, 2)$  is relatively simple, and it maps injectively (but not surjectively) to the  $\mathbb{Z}_2$ -graded moduli space. For a  $\mathbb{Z}$ -graded vector space of type  $(-2, -1, 0)$ , the map of the moduli spaces of codifferentials of order less than three *is* surjective. However, this map fails to be injective, due to a symmetry arising from an action of the symmetric group  $\Sigma_2$  on the  $\mathbb{Z}_2$ -graded space that does not appear in the  $\mathbb{Z}$ -graded case. Nevertheless, the structures of the moduli spaces are surprisingly similar.

For each space that we study, we will first establish a classification of the codifferentials of a fixed degree. Next, we will use cohomology in order to determine the nonequivalent extensions. These are the elements of our moduli space. Finally, in order to understand how the space is glued together, we will study versal deformations of each of our elements. Since our primary aims are to assemble a picture of the moduli space and to compare the  $\mathbb{Z}$ -graded and the  $\mathbb{Z}_2$ -graded moduli spaces, we will provide computational details only when they are essential. The calculations are given in more detail in [1, 5].

We would like to thank Jim Stasheff for raising this question.

## 2. THE MODULI SPACE OF $L_\infty$ ALGEBRAS

Let  $V$  be a graded vector space. If  $V$  is  $\mathbb{Z}$ -graded, then the *desuspension*  $W$  of  $V$  is defined by  $W_i = V_{i+1}$ . For a  $\mathbb{Z}_2$ -grading, this simply corresponds to reversing the parity of each element, and the resulting  $\mathbb{Z}_2$ -graded space is called the *parity reversion*. Originally,  $L_\infty$  algebras [9, 8] were defined as structures on the exterior algebra  $\bigwedge V$ , motivated by the definition of a Lie algebra, which is given by a map  $l_2 : \bigwedge^2 V \rightarrow V$ . One can translate the definition of an infinity algebra to the desuspended  $W$  picture, where it becomes a structure on the symmetric coalgebra  $S(W)$ . Although less natural, it is simpler to

express the notion of an  $L_\infty$  algebra in the  $W$  picture, because then an  $L_\infty$  algebra is simply a codifferential on the (graded) symmetric coalgebra of  $W$  [11, 10]. This is the point of view we will take in this article.

A codifferential is a coderivation whose square is zero. For  $\mathbb{Z}_2$ -graded  $L_\infty$  algebras, the codifferential is required to be odd, while for  $\mathbb{Z}$ -graded  $L_\infty$  algebras, the codifferential is required to have degree 1. (Note that it is also possible to work with the suspension, rather than the desuspension of  $V$ , in which case the codifferentials will have degree  $-1$ .) Every  $\mathbb{Z}$ -graded vector space is naturally  $\mathbb{Z}_2$ -graded, and coderivations with respect to the  $\mathbb{Z}$ -grading are also coderivations with respect to the  $\mathbb{Z}_2$ -grading. Since coderivations of degree 1 are odd, one can view  $\mathbb{Z}$ -graded  $L_\infty$  algebras as special cases of  $\mathbb{Z}_2$ -graded  $L_\infty$  algebras.

If  $d$  and  $d'$  are codifferentials on  $W$  and  $W'$  resp., then a morphism of  $L_\infty$  algebras is a coalgebra morphism  $g : S(W') \rightarrow S(W)$  satisfying  $gd' = dg$ . When  $W = W'$  and  $g$  is an automorphism of  $S(W)$ , then  $d' = g^{-1}dg$ , and we say that  $d' \sim d$ . Thus  $\text{Aut}(S(W))$  acts on the  $L_\infty$  structures, and we denote  $g^*(d) = g^{-1}dg$ . The equivalence classes of this action are called the *moduli space* of  $L_\infty$  algebras on  $W$ . The classification problem for  $L_\infty$  algebras is the study of the structure of this moduli space.

Note that for  $\mathbb{Z}$ -graded  $L_\infty$  algebras, we require a morphism of symmetric algebras to be a degree 0 map, while for  $\mathbb{Z}_2$ -graded algebras, the map is required to be even. Thus every morphism of  $\mathbb{Z}$ -graded symmetric algebras is also a morphism of  $\mathbb{Z}_2$ -graded symmetric algebras. From this it follows that two  $L_\infty$  algebras which are equivalent as  $\mathbb{Z}$ -graded  $L_\infty$  structures are equivalent as  $\mathbb{Z}_2$ -graded  $L_\infty$  algebras. Thus there is a map from the moduli space of  $\mathbb{Z}$ -graded  $L_\infty$  algebra structures to the moduli space of  $\mathbb{Z}_2$ -graded  $L_\infty$  algebras. There are fewer  $\mathbb{Z}$ -graded  $L_\infty$  structures on  $W$ , but at the same time, the notion of equivalence is more restrictive, so it is not obvious whether the map between these moduli spaces is injective or surjective. As we shall see, in general it is neither.

The main goal of this paper is to study the relationship between the moduli space of  $\mathbb{Z}$ -graded  $L_\infty$  algebras on a three-dimensional vector space  $W = W_m \oplus W_{m+1} \oplus W_{m+2}$  and the corresponding moduli space of  $\mathbb{Z}_2$ -graded  $L_\infty$  algebras. The  $\mathbb{Z}$ -graded  $L_\infty$  structures on such a vector space were studied in [3], while  $\mathbb{Z}_2$ -graded  $L_\infty$  structures on a three-dimensional space were studied in [1, 6, 5].

Define  $C^r = \text{Hom}(S^r(W), W)$  to be the space of  $r$ -cochains, that is, the cochains of exterior degree  $r - 1$ , and  $C_s^r$  to be the  $r$ -cochains of internal degree  $s$ . We will omit the adjectives internal and exterior

when the meaning is clear from the context. A  $\mathbb{Z}$ -graded codifferential  $d$  lies in  $C_1^\bullet = \prod_{r=1}^\infty C_1^r$ , so we can express  $d = d_1 + d_2 + \dots$ , where  $d_r \in C_1^r$ . There is a natural isomorphism between  $C = \prod_{r=1}^\infty C^r$  and  $\text{Coder}(S(W))$ , the space of graded coderivations of the symmetric algebra on  $W$ , so that  $C$  inherits the structure of a graded Lie algebra. Moreover,  $[C_m^k, C_n^l] \subseteq C_{m+n}^{k+l-1}$ .

The condition that  $d$  is a codifferential is that  $[d, d] = 0$ . This is equivalent to the relations

$$(1) \quad \sum_{r+s=n+1} [d_r, d_s] = 0 \quad \forall n \in \mathbb{N}.$$

If  $k$  is the least integer such that  $d_k \neq 0$ , then  $d_k$  is called the *leading term* of  $d$  and  $k$  is called the *order* of  $d$ , *i.e.*, the order of  $d$  is the degree of the least nonvanishing term in  $d$ . Note that although  $d_1$  was nonzero in many of the originally studied  $L_\infty$  structures, it is entirely possible for  $d_1$  to be zero (*e.g.*, when a classical graded Lie algebra is viewed as an  $L_\infty$  structure). Since  $d_k$  is itself a codifferential, we can classify all possible  $L_\infty$  structures by first identifying all nonequivalent codifferentials  $d_k$ , and then finding all of their nonequivalent extensions (to include those codifferentials which have higher order terms).

If we define  $D : C \rightarrow C$  by  $D(\phi) = [d_k, \phi]$ , then  $D$  is a codifferential on  $C$ ; *i.e.*,  $D^2 = 0$ . This allows us to define the cohomology  $H(d_k)$  as  $H(d_k) = \ker(D)/\text{im}(D)$ . Moreover, one can define the  $n$ -th cohomology group

$$H^n(d_k) = \ker(D : C^n \rightarrow C^{n+k-1})/\text{im}(D : C^{n-k+1} \rightarrow C^n).$$

We can re-express the relations in equation (1) in the form

$$(2) \quad D(d_{n+1}) = -\frac{1}{2} \sum_{r=k+1}^n [d_r, d_{k+n+1-r}] \quad \forall n \geq k.$$

This formula is familiar to deformation theorists, and essentially implies that  $L_\infty$  algebras are deformations of the codifferential of the leading term. In fact, one can show that if the relations hold up to order  $n$ , then the term on the right hand side in equation (2) is always a cocycle with respect to the coboundary operator  $D$ . A coderivation  $d$  which satisfies the relations equation (2) for all  $n \leq m$  is called an  $L_m$  algebra.

We can interpret the relations as saying that an  $L_m$  algebra can be extended to an  $L_{m+1}$  algebra structure precisely when the cocycle on the right hand side of equation (2) is a coboundary. We will use this observation extensively in our construction of equivalence classes of  $L_\infty$  structures.

An automorphism  $\lambda$  of  $S(W)$  is called *linear* if

$$\lambda(w_1 \cdots w_n) = \lambda(w_1) \cdots \lambda(w_n).$$

Note that this implies that  $\lambda : W \rightarrow W$  is an isomorphism. Not every automorphism is linear; in fact, if an even (degree 0 for the  $\mathbb{Z}$ -graded case) element  $\phi \in C^n$  for  $n > 1$  is considered as a coderivation of  $S(W)$ , then  $\exp(\phi) = \sum_{i=0}^{\infty} \frac{1}{i!} \phi^i$  is an automorphism of  $S(W)$ . Moreover, every automorphism  $g$  can be uniquely expressed in the form

$$g = \lambda \prod_{n=2}^{\infty} \exp(\phi_n),$$

where  $\lambda$  is a linear automorphism and  $\phi_n \in C^n$ , and every such expression yields a well defined automorphism. The term  $\lambda$  is called the *linear* part of  $g$ , and the term  $\prod_{n=2}^{\infty} \exp(\phi_n)$  is called the *higher order* part of  $g$ . We can express the action of  $g$  on codifferentials by the formula

$$(3) \quad g^* = \left( \prod_{n=\infty}^2 \exp(-\text{ad}_{\phi_n}) \right) \lambda^*,$$

which means that the action of the higher order part of  $g$  on  $C$  can be computed in terms of brackets of coderivations.

If  $d$  is a coderivation, then its automorphism group  $\text{Aut}(d)$  is the subgroup of  $\text{Aut}(S(W))$  which leaves  $d$  invariant. For a codifferential  $d_k$  of fixed degree, it is easy to see that its automorphism group is generated by the linear automorphisms  $\lambda$  fixing  $d_k$ , together with the exponentials of the even (degree 0) cocycles for the coboundary operator  $D(d_k)$  determined by  $d_k$ .

If  $d$  and  $d'$  are two equivalent codifferentials, then their leading terms are of the same degree  $k$  and linearly equivalent; *i.e.*,  $d'_k = \lambda^*(d_k)$  for some linear automorphism  $\lambda$ . Thus the first step in classifying the equivalence classes of codifferentials on  $S(W)$  is to find the linear equivalence classes of degree  $k$  codifferentials, which we call the moduli space of degree  $k$  codifferentials.

If  $d = d_k + d_l + ho$ , where  $k < l$  and  $ho$  refers to terms whose degrees are higher than  $l$ , then  $d_l$  is called the *second term* of  $d$ . It is immediate that  $D(d_l) = 0$ , so  $d_l$  is a cocycle. A codifferential is said to be in *standard form* if its second term, if any, is a nontrivial cocycle. If the second term of a codifferential is a coboundary, say  $d_l = -\frac{1}{2}D(\gamma)$  for some  $\gamma \in C^{l-k+1}$ , then  $d' = \exp(-\text{ad}_{\gamma})d$  is an equivalent codifferential whose second term has higher degree than  $l$ . Thus every codifferential is equivalent to one in standard form, and the study of nonequivalent

extensions of  $d_k$  can be reduced to the study of nonequivalent extensions of codifferentials in standard form.

If two extensions of  $d_k$  are in standard form, then the second term of each has the same degree  $l$ , and the cohomology classes of these second terms are linearly equivalent. We say that two cohomology classes  $\bar{\alpha}$  and  $\bar{\beta}$  are linearly equivalent if there is a linear automorphism  $\lambda$  of  $d_k$  such that  $\lambda^*(\bar{\alpha}) = \bar{\beta}$ . Note that this action is well defined because  $\lambda^*$  leaves  $d_k$  invariant.

If  $d = d_k + d_l$  is a codifferential and  $H^n(d_k) = 0$  for all  $n > l$ , then any extension of  $d$  is equivalent to  $d$ . This observation will be used extensively in this paper to simplify the classification of  $\mathbb{Z}$ -graded codifferentials.

### 3. DEFORMATIONS OF $L_\infty$ ALGEBRAS

For details of the theory of deformations of  $L_\infty$  algebras, see [4]. Here we recall the basic notions. Suppose that  $d_k$  is a codifferential, and  $\{\delta_i\}$  is a complete set of representatives for the cohomology with regard to  $d_k$ , in other words, a basis of a preimage of the cohomology (where  $i$  is an index for the basis, not the degree of  $\delta_i$ ). Then the *universal infinitesimal deformation*  $d^{\text{inf}}$  of  $d_k$  is given by

$$d^{\text{inf}} = d_k + \delta_i t^i,$$

where the  $t^i$ 's are parameters. (Note that we use the Einstein summation convention.) The *miniversal deformation* is the most general extension of the universal infinitesimal deformation to a formal deformation, and its primary importance is that it induces all possible formal deformations. We will use the miniversal deformation to determine the neighborhood of a given codifferential in the moduli space. In [4], miniversal deformations were constructed for  $L_\infty$  algebras of *finite type*, which includes the finite dimensional case we are considering here.

To construct the miniversal deformation, we compute  $[d^{\text{inf}}, d^{\text{inf}}]$ , and look at the higher order terms which appear. Since the bracket must be a cocycle, it can be expressed as a sum of coboundary terms and nontrivial cocycles. The coboundaries may be eliminated by adding higher order terms, so that eventually one obtains a coderivation  $d^\infty$  which has the property that its self-bracket contains no coboundaries. However, the nontrivial cocycles appearing in the bracket cannot be eliminated by this process. Accordingly, we obtain some relations on the parameters which must be satisfied if the bracket is to vanish. These relations are called the relations on the base of the versal deformation, which is the algebra  $\mathbb{K}[[t^1, t^2, \dots]]$  modulo the ideal generated by these

relations. (Here,  $\mathbb{K}$  is the base field, which we will assume later is just  $\mathbb{C}$ .)

In the process which we are describing, one can proceed level by level, constructing at each stage an  $n$ -th order deformation, for which the bracket vanishes up to order  $n + 1$ , except for terms involving the nontrivial cocycles, which give rise to  $n$ -th order relations. In the limit we obtain the miniversal deformation  $d^\infty$  and relations which are given as a formal power series.

The process described above potentially requires an infinite number of steps, although, in practice, it often terminates after a finite number of steps. It is often more practical to construct the miniversal deformation recursively as follows. Set

$$d^\infty = d + \delta_i t^i + \gamma_i x^i,$$

where  $\{\delta_i\}$  are representatives of the odd cohomology classes,  $\{\gamma_i\}$  are preimages of the even coboundaries, and  $x^i$  are formal power series in the parameters  $t^i$ . To determine the coefficients  $x^i$ , we express

$$[d^\infty, d^\infty] = \alpha_i r^i + \beta_i s^i + \tau_i y^i,$$

where  $\{\alpha_i\}$  are representatives of the even cohomology classes,  $\{\beta_i\}$  is a basis of the even coboundaries, and  $\alpha_i, \beta_i, \tau_i$  are a basis of the even cochains. It follows from [4] that  $s^i = 0$  for all  $i$ ,  $r^i$  are formal power series in the parameters  $t^i$ , and  $y^i = 0 \pmod{(r^1, \dots)}$ . Moreover, the equations  $s^i = 0$  can be solved to obtain the expressions for  $x^i$  as formal power series in the parameters  $t^i$ .

Suppose that we have constructed a miniversal deformation

$$(4) \quad d^\infty = d + \delta_i t^i + \dots .$$

Suppose some set of values of the parameters  $t^i$  satisfy the relations on the base. The codifferential obtained by substituting these values into the formula for the miniversal deformation is a well-defined codifferential in the moduli space. We can think of the values of the parameters which satisfy the relations as determining a variety in the  $(t^i)$  space, and the miniversal deformation as a description of the elements in the moduli space which are near to it. Thus the miniversal deformation provides information about how the moduli space is glued together.

A one parameter family  $d(t)$  of deformations of  $d_k$  is a curve in the  $(t^i)$  space such that  $d(0) = d_k$ . It can happen that there is a codifferential  $d'$  such that  $d(t) \sim d'$  whenever  $t \neq 0$ . In this case, the one-parameter family of deformations is called a *jump deformation*. Otherwise,  $d(t)$  is called a family of smooth deformations. It is interesting to note that jump deformations are one way transformations; if  $d$  has a jump

deformation to  $d'$ , then there is never any jump deformation in the opposite direction.

Suppose that there is a smooth family  $d(t)$  of deformations of  $d$ . Then this family is said to factor through a jump deformation from  $d$  to  $d'$  if there is a family  $d'(t)$  of deformations of  $d'$  such that  $d'(t) \sim d(t)$  whenever  $t \neq 0$ . In this case, we do not consider the deformations  $d(t)$  to belong to the same *stratum* of the moduli space of  $d$ , which is determined by the smooth families of deformations of  $d$  which do not factor through jump deformations. We will not explain here why the notion of stratum is well defined, but use this notion only for describing the moduli space. The pattern we will observe in this paper is that the moduli space has a decomposition in terms of these strata, and that each stratum is an orbifold. The smooth deformations which do not factor through jump deformations determine how a stratum is glued together, and the jump deformations determine how the strata are glued to each other.

#### 4. NOTATION

It is customary when working with  $\mathbb{Z}_2$ -graded spaces to list the even elements of a basis first, and we will follow this convention even though it means that our basis elements will not be listed in the consecutive order that is natural from the  $\mathbb{Z}$ -graded point of view. In order to express our codifferentials in a compact form, we will use the following notation. Let  $\{e_1, e_2, e_3\}$  be a fixed basis of  $W$ . Then  $\varphi_k^{i_1 i_2 i_3} \in \text{Hom}(S^{i_1+i_2+i_3}(W), W)$  is defined by

$$\varphi_k^{i_1 i_2 i_3}(e_1^{j_1} e_2^{j_2} e_2^{j_2}) = i_1! i_2! i_3! \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \delta_{j_3}^{i_3} e_k.$$

The factorials in the above definition give rise to simpler formulas for the brackets of coderivations. In order to make it easier to distinguish between even and odd coderivations, if  $\varphi_l^{ijk}$  is odd, we will denote it as  $\psi_l^{ijk}$  instead. In Section 2, we defined  $C^r = \text{Hom}(S^r(W), W)$  to be the space of  $r$ -cochains, and  $C_s^r$  to be the cochains of internal degree  $s$ . Then  $\varphi_k^{i_1 i_2 i_3} \in C_s^{i_1+i_2+i_3}$ , where  $s = |e_k| - (i_1|e_1| + i_2|e_2| + i_3|e_3|)$  is the difference of the degrees of the output and input of  $\varphi_k^{i_1 i_2 i_3}$  (*i.e.*, the degree of  $\varphi_k^{i_1 i_2 i_3}$  as a map).

From now on, let us assume that the base field is  $\mathbb{C}$ . Note that in our definition of exponentials of coderivations, we implicitly have been assuming that the field has characteristic zero.

### 5. CODIFFERENTIALS ON A 1|2-DIMENSIONAL SPACE

If  $W$  is of the form  $W = W_{2x-1} \oplus W_{2x} \oplus W_{2x+1}$ , then as a  $\mathbb{Z}$ -graded space, it has type  $(2x-1, 2x, 2x+1)$ , but its dimension is always  $1|2$  as a  $\mathbb{Z}_2$ -graded space. In order to list the even basis elements first, we set  $|e_1| = 2x$ ,  $|e_2| = 2x-1$  and  $|e_3| = 2x+1$ . Then we decompose the space of cochains as follows.

$$C_{2s}^r = \begin{cases} \langle \varphi_1^{r,0,0}, \varphi_1^{r-2,1,1}, \varphi_2^{r-1,1,0}, \varphi_3^{r-1,0,1} \rangle, & s = -x(r-1) \\ \langle \varphi_2^{r-1,0,1} \rangle & s = -x(r-1)-1 \\ \langle \varphi_3^{r-1,1,0} \rangle & s = -x(r-1)+1 \end{cases}$$

$$C_{2s+1}^r = \begin{cases} \langle \psi_1^{r-1,1,0}, \psi_3^{r,0,0}, \psi_3^{r-2,1,1} \rangle & s = -x(r-1) \\ \langle \psi_1^{r-1,0,1}, \psi_2^{r,0,0}, \psi_2^{r-2,1,1} \rangle & s = -x(r-1)-1 \end{cases}$$

Note that  $C_1^1$  has dimension 2 for all  $x$ . When  $x = -1$ ,  $C_1^2$  has dimension 3. Otherwise,  $C_1^r = 0$  for all  $r > 1$ , unless  $x = 0$ , in which case  $C_1^r$  has dimension 3 for all  $r > 1$ . We study the most important case, when  $x = 0$ , but the case  $x = 1$  has some interesting features as well.

**5.1. Codifferentials on  $\mathbf{W} = \mathbf{W}_{-1} \oplus \mathbf{W}_0 \oplus \mathbf{W}_1$ .** The basis elements of  $W = W_{-1} \oplus W_0 \oplus W_1$  have degrees  $|e_1| = 0$ ,  $|e_2| = -1$  and  $|e_3| = 1$ , and the space of cochains has the following decomposition.

$$\begin{aligned} C_{-2}^r &= \langle \varphi_2^{r-1,0,1} \rangle \\ C_{-1}^r &= \langle \psi_1^{r-1,0,1}, \psi_2^{r,0,0}, \psi_2^{r-2,1,1} \rangle \\ C_0^r &= \langle \varphi_1^{r,0,0}, \varphi_1^{r-2,1,1}, \varphi_2^{r-1,1,0}, \varphi_3^{r-1,0,1} \rangle \\ C_1^r &= \langle \psi_1^{r-1,1,0}, \psi_3^{r,0,0}, \psi_3^{r-2,1,1} \rangle \\ C_2^r &= \langle \varphi_3^{r-1,1,0} \rangle \end{aligned}$$

We shall refer to the bases above as the standard bases of these spaces. Note that for  $r = 1$ , there are no terms with  $r-2$  in their superscripts. Since a codifferential for a  $\mathbb{Z}$ -graded  $L_\infty$  algebra must have degree 1, any codifferential of order  $k$  is of the form

$$d = \psi_1^{k-1,1,0}a + \psi_3^{k,0,0}b + \psi_3^{k-2,1,1}c.$$

Since  $[d, d] = \varphi_3^{2k-2,1,0}2b(ka - c)$ , it follows that  $d$  is a codifferential precisely when  $b = 0$  or  $c = ka$ . We will now show that any such codifferential is equivalent to one of the three types below:

$$(5) \quad \begin{aligned} d_k(\lambda : \mu) &= \psi_1^{k-1,1,0}\lambda + \psi_3^{k-2,1,1}\mu, \\ d_k^* &= \psi_3^{k,0,0}, \\ d_k^\sharp &= \psi_1^{k-1,1,0} + \psi_3^{k,0,0} + \psi_3^{k-2,1,1}k. \end{aligned}$$

If  $b = 0$ , then we obtain the codifferential  $d_k(\lambda : \mu) = \psi_1^{k-1,1,0}\lambda + \psi_3^{k-2,1,1}\mu$ . A diagonal linear automorphism takes  $d_k(\lambda : \mu)$  to a multiple of itself, so that the codifferentials of this type are parameterized by  $(\lambda : \mu) \in \mathbb{P}^1(\mathbb{C})$ . If  $b \neq 0$ , we must have

$$d = \psi_1^{k-1,1,0}\lambda + \psi_3^{k,0,0}\mu + \psi_3^{k-2,1,1}k\lambda.$$

This time, if  $g = \text{diag}(1, q, r)$  is the diagonal linear transformation with  $1, q$  and  $r$  on the main diagonal, then

$$g^*(d) = \psi_1^{k-1,1,0}q\lambda + \psi_3^{k,0,0}\mu r^{-1} + \psi_3^{k-2,1,1}qk\lambda,$$

so that if neither  $\lambda$  nor  $\mu$  vanishes the codifferential is equivalent to  $d_k^\sharp = \psi_1^{k-1,1,0} + \psi_3^{k,0,0} + \psi_3^{k-2,1,1}k$ . When  $\lambda = 0$ , we obtain the codifferential  $d_k^* = \psi_3^{k,0,0}$ . The case  $\mu = 0$  is simply a duplicate of the codifferential  $d_k(\lambda : k\lambda)$ .

**5.2. Extensions of the Codifferentials.** We now study the deformations and extensions of each of these types of codifferentials. We will show in sections 5.7 and 5.8 that the cohomology  $H_1^n$  vanishes for  $n > k$  for the codifferentials  $d_k^*$  and  $d_k^\sharp$ . Because of this, they have no nonequivalent extensions. In general, this will be true for the codifferentials  $d_k(\lambda : \mu)$ , except for some special cases.

Let  $d = d_k(\lambda : \mu)$ . The matrix of  $D : C_1^l \rightarrow C_2^{k+l-1}$  is  $[0 \ \lambda l - \mu \ 0]$ , so unless  $\mu = \lambda l$ , the dimension of the space  $Z_1^l$  of  $l$ -cocycles is 2. Thus the codifferential  $d_k(1 : l)$  is special when  $l$  is any positive integer, because the dimension of the space of cocycles is one higher than usual.

The matrix of  $D : C_0^l \rightarrow C_1^{k+l-1}$  is  $\begin{bmatrix} \lambda(k-l-1) & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \\ \mu(k-2) & 0 & \mu & -\lambda(l-1) \end{bmatrix}$ . When  $\lambda = 0$  or  $k = l$ , the rank of this matrix is equal to 1; otherwise the rank is 2. Thus generically, when  $l > k$  we have  $H_1^n(d_k(\lambda : \mu)) = 0$  for all  $n > k$ . The only cases where nontrivial extensions occur are for the codifferentials  $d_k(0 : 1)$  and  $d_k(1 : l)$ , when  $l > k$ .

**5.2.1. Extensions of  $d_k(0 : 1)$ .** In this case we have

$$\begin{aligned} H_1^1 &= \langle \psi_1^{0,1,0} \rangle \\ H_1^n &= \langle \psi_1^{n-1,1,0}, \psi_3^{n-2,1,1} \rangle, \quad 1 < n < k \\ H_1^n &= \langle \psi_1^{n-1,1,0} \rangle, \quad n \geq k. \end{aligned}$$

We can extend  $d_k(0 : 1)$  to the codifferential  $\psi_3^{k-2,1,1} + \psi_1^{l-1,1,0}x$ . When  $x \neq 0$ , by applying a linear automorphism, we can transform this codifferential to one of the same form with  $x = 1$ . Thus, up to equivalence,

for each  $l > k$ , we obtain an extension

$$(6) \quad d_{k,l} = \psi_3^{k-2,1,1} + \psi_1^{l-1,1,0}.$$

It can be shown that  $H^n(d_{k,l}) = 0$  for  $n > l$ , unless  $n = 2l - k$ . Since the cohomology does not vanish completely, it is natural to suspect that one can add a nontrivial term of degree  $2l - k$ . In fact, any extension of  $d_{k,l}$  must be of the form

$$d^e = d_{k,l} + \psi_1^{n-1,1,0}\alpha + \psi_3^{n-2,1,1}\beta + ho,$$

since the third nonvanishing term in such an extension has to be a cocycle with respect to  $d_k$ . Applying the automorphism  $\exp(-\varphi_2^{n-k,1,0}\beta)$  to  $d_e$  will replace the term  $\beta$  term in  $d^e$  with higher order terms, so we can assume it is zero. If  $\eta = \varphi_1^{n-l+1,0,0} - \varphi_2^{n-l,1,0}(k-2)$ , then  $D(\eta) = 0$  and  $[\psi_1^{l-1,1,0}, \eta] = -\psi_1^{n-1,1,0}(n+k-2l)$ , so as long as  $n \neq 2l - k$ , we can eliminate the  $\alpha$  term in  $d^e$  as well. We cannot eliminate the term in degree  $2l - k$ , which is exactly the degree in which the cohomology does not vanish. Thus we obtain a nontrivial extension of  $d_{k,l}$  given by

$$(7) \quad d_{k,l}(\alpha) = d_{k,l} + \psi_1^{2l-k-1,1,0}\alpha.$$

Moreover, a diagonal automorphism  $g = \text{diag}(p, q, t)$  leaves  $d_{k,l}$  invariant precisely when  $p^{l-k} = 1$ , and in this case  $g$  leaves  $\psi_1^{2l-k,1,0}$  invariant. From this it follows that the extensions  $d_{k,l}(\alpha)$  are nonequivalent for different values of  $\alpha$ . Thus we obtain a family of nonequivalent extensions of  $d_k(0 : 1)$ .

But this is the end of the story, because it can be shown that any extension of  $d_{k,l}(\alpha)$  is equivalent to it. Note this is not surprising, because  $H_1^n(d_{k,l}(\alpha)) = 0$  for  $n > 2l - k$ .

**5.2.2. Extensions of  $d_k(1 : l)$ , where  $l > k$ .** Since  $\psi_3^{l,0,0}$  is a nontrivial cohomology class, we obtain a nontrivial extension of  $d_k(1 : l)$  by adding a multiple of this cocycle, and by the usual argument, we can assume the coefficient of the added term is one. Thus we obtain the nontrivial extension

$$(8) \quad d_{k,l}^\sharp = \psi_1^{k-1,1,0} + \psi_3^{l,0,0} + \psi_3^{k-2,1,1}l.$$

The matrix of  $\text{ad}_{\psi_3^{l,0,0}} : C_0^n \rightarrow C_1^{l+n-1}$  is  $\begin{bmatrix} 0 & 1 & 0 & 0 \\ l & 0 & 0 & -1 \\ 0 & l & 0 & 0 \end{bmatrix}$ , and the matrix of  $\text{ad}_{\psi_3^{l,0,0}} : C_1^n \rightarrow C_2^{l+n-1}$  is  $\begin{bmatrix} l & 0 & -1 \end{bmatrix}$ . It follows that  $d_{k,l}^\sharp$ -cocycles of degree larger than  $l$  have the form  $\psi_1^{n-1,1,0} + \psi_3^{n-1,1,1}l$ , and these terms are always  $d_{k,l}^\sharp$ -coboundaries. Thus  $H^n(d_{k,l}^\sharp) = 0$  vanishes for  $m \geq l$ , which suggests that the extensions  $d_{k,l}^\sharp$  are the only nontrivial extensions of

$d_k(1 : l)$ . In fact, if we write an extension of  $d_{k,l}^\sharp$  by adding terms of degree  $n$  or higher, then it has to be of the form

$$(d_{k,l}^\sharp)^e = d_{k,l}^\sharp + \psi_1^{n-1,1,0}\alpha + \psi_3^{n-2,1,1}\beta + ho,$$

since the degree  $n$  term must be a  $d_k(1 : l)$ -cocycle. But the term of degree  $n$  can be eliminated by applying an exponential, so we don't obtain any higher order nonequivalent extensions. Note that if  $l \leq k$ , then  $d_k(1 : l)$  has no nontrivial extensions.

**5.3. Deformations of the nonequivalent codifferentials.** As a result of our study of extensions, we can now give a complete classification of all nonequivalent codifferentials on  $W$ . Let us relabel  $d_k^\sharp$  as  $d_{k,k}^\sharp$ , because it fits nicely into the pattern  $d_{k,l}^\sharp$ , and we shall see later that it deforms in a similar manner to elements of this form. The complete list of nonequivalent codifferentials is given by

$$(9) \quad \begin{aligned} d_k(\lambda : \mu) &= \psi_1^{k-1,1,0}\lambda + \psi_3^{k-2,1,1}\mu \\ d_k^* &= \psi_3^{k,0,0} \\ d_{k,l}(\alpha) &= \psi_3^{k-2,1,1} + \psi_1^{l-1,1,0} + \psi_1^{2l-k-1,1,0}\alpha, \quad k \leq l \\ d_{k,l}^\sharp &= \psi_1^{k-1,1,0} + \psi_3^{k-2,1,1}l + \psi_3^{l,0,0}, \quad k \leq l. \end{aligned}$$

We now construct miniversal deformations for each of these codifferentials and then determine which codifferentials arise as we substitute values of the parameters into the miniversal deformation. This tells us how the codifferentials deform locally, and gives us the gluing relations on the moduli space.

**5.3.1. Deformations of  $d_k(\lambda : \mu)$ : Generic Case.** In this section, we suppose that  $\lambda \neq 0$  and  $\mu/\lambda \notin \mathbb{Z}_+$ . Then the cohomology is given by

$$\begin{aligned} H_1^1 &= \langle \psi_1^{0,1,0} \rangle \\ H_1^n &= \langle \psi_1^{n-1,1,0}, \psi_3^{n-2,1,1} \rangle \quad 1 < n < k \\ H_1^k &= \langle \psi_1^{k-1,1,0} \rangle \\ H_1^n &= 0, \quad n > k. \end{aligned}$$

As a consequence, the universal infinitesimal deformation is given by

$$\begin{aligned} d_k^\infty(\lambda : \mu) &= d_k(\lambda : \mu) + \psi_1^{m-1,1,0}s_m + \psi_3^{n-2,1,1}t_n, \quad 1 \leq m < k, 1 < n \leq k \\ &= d_k(\lambda : \mu + t_n) + \sum_{n=1}^{k-1} d_n(s_n : t_n). \end{aligned}$$

Furthermore, this deformation is miniversal since its self-bracket vanishes. In other words, the miniversal deformation is given by the same formula above, interpreting the parameters as lying in the formal power

series algebra  $\mathbb{C}[[s_n, t_n]]$ . Because the bracket vanishes, there are no relations on the base of the miniversal deformation.

If we substitute some values for the parameters in the formula above, we want to determine the equivalence class of the resulting codifferential. Let  $M$  be the least value of  $m$  so that  $s_m \neq 0$ , if it exists, and similarly, let  $N$  be the least value of  $n$  so that  $t_n$  does not vanish. If  $s_m$  vanishes for all  $m$  and  $N < k$ , then the deformation is equivalent to  $d_{N,k}(-\frac{t_k+\mu}{\lambda})$ . When  $s_m$  vanishes for all  $m$  and  $N = k$ , the deformation is equal to  $d_k(\lambda : \mu + t_n)$ , which lies in the family.

If  $M \leq N$  or  $t_n = 0$  for all  $n$ , then the deformation is equivalent to  $d_M(s_M : t_M)$ , since all the higher order terms are coboundaries with respect to the leading term  $d_M(s_M : t_M)$  in  $d^\infty$ . Since the coordinates  $(s_M : t_M)$  are projective, there are deformations to  $d_M(s_M : t_M)$  for arbitrarily small values of the parameters, which means that the deformations to  $d_M(s_M : t_M)$  are jump deformations.

If  $M > N$ , then the deformation is equivalent to  $d_{N,M}(\beta)$ , where  $\beta$  depends rationally on the parameters  $t_n$  and  $s_m$ . The denominator is  $s_M^{M-N+1}$ , and the numerator is a polynomial in  $s_M$  of degree  $M - N$ , with coefficients given by polynomials in the parameters  $t_n$  and  $s_m$  for  $m > M$ , with no constant term in any of the coefficients. For example, if  $k = 5$ ,  $N = 3$  and  $M = 5$ , then  $\beta = \frac{-s_5^2 t_5 + (\lambda t_3 + s_6 t_4) s_5 - s_6^2 t_3}{s_5^3}$ .

Therefore, every possible value of  $\beta$  arises along curves with arbitrarily small values of the parameters. As a consequence, there are jump deformations to  $d_{N,M}(\beta)$  for all  $\beta$ .

The only deformations which do not occur as jump deformations are the deformations  $d_k(\lambda : \mu + t_n)$  along the family. Thus the pattern is that we deform along the family  $d_k(\lambda : \mu)$  of the same order of codifferential, and to lower order objects of the form  $d_n(\alpha, \beta)$  for all  $(\alpha : \beta)$ , as well as to all  $d_{n,m}(\beta)$  for  $n < k$  and  $m < k$ .

#### 5.4. Deformations of $d_k(0 : 1)$ .

The miniversal deformation is

$$d_k^\infty(0 : 1) = d_k(0 : 1) + \psi_1^{m-1,1,0} s_m + \psi_3^{n-2,1,1} t_n, \quad 1 \leq m < \infty, 1 < n < k.$$

The deformation picture is the same as in the generic case, with the following exceptions. If  $s_m = 0$  for all  $m$ , then the deformation is equivalent to  $d_N(0 : 1)$ . The new case of interest is when  $M > k$ . Then the deformation is equivalent to  $d_{N,M}(\beta)$ , where  $\beta$  is a rational function with denominator  $s_M^{M-N+1}$ , the numerator is a polynomial in  $s_M$  of degree less than  $M - N + 1$ , with coefficients given by polynomials without constant terms in the parameters  $t_n$  and  $s_m$  for  $m > M$ . For example, if

$$k = 7, N = 6, \text{ and } M = 8, \text{ then } \beta = \frac{s_{10}t_6s_8 + s_9t_7s_8 + s_9\mu s_8 - s_9^2t_6}{s_8^3}.$$

As a consequence, we see that there are jump deformations to  $d_{N,M}(\beta)$  for all values of  $\beta$ . Note that the deformation along the family is given by  $d_k(s_k : 1)$  instead of  $d_k(\lambda : \mu + t_k)$ , and this deformation is the only one which is not attainable as a jump deformation.

**5.5. Deformations of  $d_{k,l}(\alpha)$ .** The miniversal deformation is given by

$$d_{k,l}^\infty(\alpha) = d_{k,l}(\alpha + s_{2l-k}) + \sum_{n=1}^{l-1} d_n(s_n, t_n),$$

where  $t_1 = 0$  and  $t_n = 0$  for  $n > k$ . The picture of the deformations is similar to that of  $d_k(0 : 1)$ . When  $M = k$  and  $N = k$  or  $N$  does not exist, we get the deformation  $d_k(s_k : t_k + 1)$ . Otherwise, when  $M < k$  and  $M \leq N$  or  $N$  does not exist, the deformation is equivalent to  $d_M(s_M : t_M)$ . Also, as we explain below, deformations of the form  $d_{N,M}(\beta)$  arise only when  $N < M < l$ .

When  $N < M \leq k$ , or when  $N \leq k$  and  $M = k + 1$  we get jump deformations to  $d_{N,M}(\beta)$  for all values of  $\beta$ . We also get jump deformations to  $d_{k,k+1}(\beta)$  for all  $\beta$ , which corresponds to the case when  $N$  does not exist, and  $M = k + 1$ . If  $s_m = 0$  for all  $m < 2l - k$  then the deformation is equivalent to  $d_{N,l}((1 + t_N)(\alpha + s_{2l-k}))$  if  $N < k$ , and to  $d_{k,l}(\alpha + s_{2l-k})$ , when  $N$  does not exist. These are smooth deformations to a neighborhood of  $d_{N,l}(\alpha)$  (or  $d_{k,l}(\alpha)$ ), not jump deformations.

When  $N < k \leq M$  and  $M \neq k + 1$ , or  $N = k$  and  $M > k + 1$ , the deformation is again of the form  $d_{N,M}(\beta)$ , where again,  $\beta$  is a rational function of the parameters with denominator  $s_M^{M-N+1}$  and the numerator is a polynomial in  $s_M$  of degree  $M - N$  or less, but this time, one of the coefficients of the polynomial is simply  $t_k + 1$ . The case where  $N$  does not exist is similar, but this time it is to  $d_{k,M}(\beta)$  and one of the coefficients of the polynomial is 1. This means that the numerator is bounded away from zero for small values of the parameters, while the denominator is close to zero. For example, if  $k = 6$ ,  $l = 9$ ,  $N = 4$  and  $M = 8$ , then the deformation is equivalent to  $d_{4,8} \left( \frac{(\alpha + s_{12})t_4s_8^3 - (t_6 + 1)s_8^2 + s_8t_5 - t_4}{s_8^5} \right)$ . As a consequence, we obtain smooth deformations to  $d_{N,M}(\beta)$  in a neighborhood of  $\beta = \infty$ .

What is the meaning of the deformations which occur only for large values of  $\beta$ ? If  $l > k + 1$ , then we can think of the  $d_{k,l}(\alpha)$  as points lying in an affine plane in  $\mathbb{P}^1(\mathbb{C})$ , parameterized by the coordinate  $\alpha$ , with  $d_{k,l+1}(\beta)$  glued in as the point at infinity. Furthermore, each of these families  $d_{k,l}(\alpha)$  is glued in as the point at infinity for each smaller

one (*i.e.*, for each  $d_{m,n}(\beta)$  with  $m < k \leq n < l$ ). Thus we can always deform to lower order objects on the sequence  $d_{k,l}(\alpha)$  to points near infinity, that is, deformations with large values of  $\beta$ . The exception is that  $d_{k,k+1}(\alpha)$  has jump deformations from  $d_{k,l}(\beta)$  for all values of  $\alpha$  and  $\beta$  and all  $l > k + 1$ .

**5.6. Deformations of  $d_k(1 : l)$ .** The universal infinitesimal deformation is

$$d_k^{\text{inf}}(1 : l) = d_k(1 : l + t_k) + \sum_{n=1}^{k-1} d_n(s_n, t_n) + \psi_3^{l,0,0} r,$$

where  $t_1 = 0$ . This time, the self bracket does not vanish. In fact,

$$\frac{1}{2}[d_k^{\text{inf}}(1 : l), d_k^{\text{inf}}(1 : l)] = \sum_{n=1}^k r(ls_n - t_n).$$

Some of these terms are coboundaries, so we can add some preimages of coboundaries to cancel them. The terms which are not coboundaries will lead to the relations on the base of the miniversal deformation. We will need to add some terms to the infinitesimal deformation to get the miniversal deformation. These terms can be determined recursively, as follows. Suppose that

$$d_k^\infty(1 : l) = d_k^{\text{inf}}(1 : l) + \sum_{m=1}^{l-1} \psi_3^{m,0,0} x_m,$$

in other words, it is obtained by adding terms involving the preboundaries  $\psi_3^{m,0,0}$ . Then we have

$$\begin{aligned} \frac{1}{2}[d^\infty, d^\infty] &= \sum_{m=1}^{l-1} \sum_{n=1}^k \varphi_3^{k+m-2,1,0} x_m(m-l) + \varphi_3^{l+n-2,1,0} r(ls_n - t_n) \\ &\quad + \varphi_3^{n+m-2,1,0} x_m(ms_n - t_n). \end{aligned}$$

These equations can be solved for the coefficients  $x_m$ , which will be given by rational expressions in the parameters  $r$ ,  $s_n$  and  $t_n$ . In fact  $x_{l-1} = \frac{r(ls_{k-1} - t_{k-1})}{1+t_k}$ , and the others can be found by a downward recursion. Note that as a formal power series in the parameters,  $x_{l-1}$  is of order 2; the other  $x$ 's will also be of at least this order.

There are  $k+l-1$  terms in the self-bracket of  $d^\infty$ , of which the terms of degrees 1 through  $k-1$  and degree  $k+l-1$  correspond to nontrivial cocycles. The other  $l-1$  terms are used to solve for the  $x_m$  recursively.

As a simple example, let  $l = 2$  and  $k = 3$ . Then we need to add only  $x_1 = \frac{r(2s_2 - t_2)}{1+t_3}$ . There are 3 relations on the base,

$$rt_3 = 0, \quad 2rs_1 + x_1(s_2 - t_2) = 0, \quad x_1s_1 = 0.$$

These relations must be satisfied by any codifferential which is obtained from the miniversal deformation by substituting values in for the parameters, and any substitution for the parameters satisfying these relations will determine an actual codifferential. To understand the information encoded in the miniversal deformation, we need to analyze the equivalence classes of the resulting codifferentials we obtain in this manner. For our example, we see that a deformation with  $x_1 \neq 0$  will be equivalent to  $d_1^*$ . If  $x_1 = 0$ , and  $r = 0$ , the deformation is equivalent to either  $d_1(1 : 0)$ ,  $d_2(s_2 : t_2)$  or  $d_3(1 : l + t_3)$ . On the other hand, if  $r \neq 0$ , then  $s_1 = t_3 = 0$  and  $2s_2 = t_2$ . This give rise to  $d_{2,2}^\sharp$ .

The general pattern is that  $d_n^*$  will arise as a deformation of  $d_k(1 : l)$ ; in addition,  $d_{n,m}^\sharp$  will appear when  $n \leq k$  and  $m \leq l$ .

### 5.7. Deformations of $d_{k,l}^\sharp$ .

The cohomology is given by

$$\begin{aligned} H_1^1 &= \langle \zeta = \psi_1^{0,1,0}(k-1) + \psi_3^{1,0,0}l \rangle \\ H_1^n &= \langle d_n(1 : l), d_{n,n+l-k} \rangle, 1 < n < k \\ H_1^n &= 0, n \geq k \end{aligned}$$

The universal infinitesimal deformation is given by

$$(d_{k,l}^\sharp)^{\text{inf}} = d_{k,l}^\sharp + \zeta r + d_m(1 : k)s^m + d_{n,n+l-k}t^n, \quad 1 < m < k, 1 < n < k$$

The term  $\zeta$  coming from  $H^1$  does not fit the nice pattern we have been observing, and indeed, when we compute the self-bracket of  $(d_{k,l}^\sharp)^{\text{inf}}$ , the lowest degree term is multiplied by  $2lr^2(k-1)(l-k+1)$ , which means that in any actual deformation, we must have  $r = 0$ . In classical language, this means that  $d_{k,l}^\sharp + \zeta r$  does not extend to a second order deformation. Let us assume that  $r = 0$  and then compute the self bracket. We obtain

$$\frac{1}{2}[(d_{k,l}^\sharp)^{\text{inf}}, (d_{k,l}^\sharp)^{\text{inf}}] = \varphi_3^{n-2+l-k+m,1,0}s^mt^n(n-k).$$

It is easy to see that  $d_{k,l}^\sharp$  deforms to the lower order codifferentials  $d_{n,n+l-k}^\sharp$  as well as the codifferentials  $d_{n,l}^\sharp$ , when  $n = 2 \dots k-1$ . What is less obvious is that we also may obtain codifferentials of the form  $d_n^*$  for  $n = 1 \dots k-1$  and codifferentials of the form  $d_{n,m}^\sharp$  for  $1 < n \leq m < k$ .

In general, it is complicated to describe precisely which lower order codifferentials will arise. For example, if  $k = 3$  and  $l = 4$ , the codifferentials  $d_1^*$  and  $d_{2,2}^\sharp$  occur, but not  $d_2^*$ . Thus, all deformations are to codifferentials of the type listed above, but not all of the possibilities actually occur for a particular codifferential.

**5.8. Deformations of  $d_k^*$ .** The cohomology is somewhat different for this codifferential. We have

$$\begin{aligned} H_1^1 &= \langle \psi_3^{1,0,0} \rangle \\ H_1^n &= \langle \psi_3^{n,0,0}, \psi_1^{n-1,1,0} + \psi_3^{n-2,0,1} n \rangle & 1 < n < k \\ H_1^k &= \langle \psi_1^{k-1,1,0} + \psi_3^{k-2,0,1} k \rangle \\ H_1^n &= 0, & n > k. \end{aligned}$$

As a consequence, we see that there are no extensions of  $d_k^*$ , and that it deforms into the codifferential  $d_{k,k}^\sharp$ , of the same order. The universal infinitesimal deformation is given by

$$(d_k^*)^{\text{inf}} = d_k^* + \psi_3^{m,0,0} s_m + (\psi_3^{n-1,1,0} + \psi_3^{n-2,1,1} k) t_n, \quad 1 \leq m < k, 1 < n \leq k.$$

We compute the self bracket

$$(10) \quad \frac{1}{2}[(d_k^*)^{\text{inf}}, (d_k^*)^{\text{inf}}] = \varphi_3^{m+n-2,1,0} s_m t_n (m - k).$$

Since coboundary terms occur in the bracket, we have to add some terms of the form  $\psi_1^{r-1,1,0} x_r$ , in fact, exactly  $k-1$  of them to correspond to the  $k-1$  coboundary terms in the relations above. Note that in the end, there will be  $k-1$  relations on the base of the miniversal deformation. In this case, we do not need to solve all of the equations in order to determine what codifferentials arise as deformations. It is clear that we can deform to  $d_m^*$  for  $1 \leq m < k$  as well as to  $d_{n,k}^\sharp$  for  $1 < n \leq k$ . But it may occur that the lowest  $m$  term added is of larger degree than the lowest  $n$  term. In this case, we would expect that the deformation is equivalent to  $d_{n,m}^\sharp$ . However, this would not be true without the addition of the  $x_r$  terms, which fortunately are added in the versal deformation. These terms are precisely what is necessary in order for the deformed codifferential to be equivalent to  $d_{n,m}^\sharp$ . Thus, we can also deform to the codifferentials  $d_{n,m}^\sharp$  for  $1 < n \leq m < k$ .

**5.9. Description of Figure 1.** The upper large sphere represents  $d_k(\lambda : \mu)$ , and the large sphere below it is  $d_{k-1}(\lambda : \mu)$ . Although the diagram only shows two distinct strata (at levels  $k$  and  $k-1$ ), there is actually an infinitely long chain of them. We will now describe the  $k^{\text{th}}$  stratum (shown in the upper half of the picture) in detail.

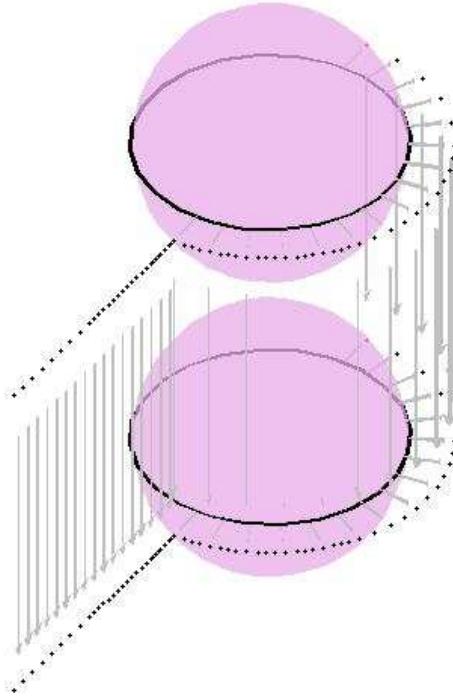


FIGURE 1. **The moduli space of codifferentials on  $\mathbf{W}_{-1} \oplus \mathbf{W}_0 \oplus \mathbf{W}_1$ .** This picture is explained in section 5.9.

The large sphere,  $d_k(\lambda : \mu)$ , can be thought of as the complex projective space  $\mathbb{P}^1(\mathbb{C})$ . The point on the equator which has a long spike extending above it is  $d_k(0 : 1)$ , the point at infinity. The equator also includes all points  $d_k(1 : l)$ . The points above the equator (except for the long spike) represent  $d_{k,l}^\sharp$  (for each  $l \in \mathbb{Z}$ ). Recall that when  $l \in \mathbb{Z}$ ,  $d_k(1 : l)$  has a jump deformation to  $d_{k,l}^\sharp$ , which is represented by the arrows radiating out from the equator. Note that there are also many other deformations of this sort which are not explicitly shown in our diagram (for the sake of simplicity). Specifically,  $d_k(1 : l)$  has jump deformations to  $d_{m,n}^\sharp$  for all  $m \leq k$  and  $n \leq l$ .

The long spike above the point at infinity of the large sphere represents the family  $d_{k,l}(\alpha)$ . The “smallest” element of the family on level  $k$

is  $d_{k,k+1}(\alpha)$ , which is on the end of the spike which is farthest from the large sphere  $d_k(\lambda : \mu)$ . The entire spike consists of an infinite sequence,  $\{d_{k,l}(\alpha) | l > k\}$ , which gets closer to the sphere as  $l$  increases.

Since each member of the family  $\{d_{k,l}(\alpha) | l > k\}$  is parameterized by a complex number  $\alpha$  and has a point at infinity, we can visualize each  $d_{k,l}(\alpha)$  as a sphere. Specifically, we can think of  $d_{k,l+1}(0)$  as the point at infinity on  $d_{k,l}(\alpha)$ , since  $d_{k,l+1}(0)$  deforms smoothly to  $d_{k,l}(\beta)$  in a neighborhood of  $\beta = \infty$ . From this point of view, one could visualize  $\{d_{k,l}(\alpha) | l > k\}$  as an infinitely long string of pearls, as indicated by the diagram. However, this pretty picture is an oversimplification, since we can actually deform smoothly to  $d_{k,l}(\beta)$  in a neighborhood of  $\beta = \infty$  from  $d_{m,n}(\alpha)$  for all  $k \leq m \leq l < n$ , and for all  $\alpha$ .

From each  $d_{k,l}(\alpha)$ , there is a smooth deformation to a neighborhood of each  $d_{n,l}(\alpha)$  “below” it (*i.e.*, for each  $n < k$ ), represented by the downward arrows descending from the spike. In the interest of keeping the diagram uncluttered, there are many other deformations which are not explicitly represented by arrows, but which are described in the next paragraph.

From each point on  $d_k(\lambda : \mu)$ , there is a jump deformation to each “lower”  $d_{m,n}(\beta)$  (in other words, for each  $m < n \leq k$ ). Additionally, there is a jump deformation from  $d_{k,l}(\alpha)$  to each  $d_m(\alpha : \beta)$  which is at the same level or lower, *i.e.*, for each  $m \leq k$ . From each point  $d_k(\lambda : \mu)$  on the large sphere, there exist smooth deformations to every other point  $d_k(\alpha : \beta)$  on the same sphere. There also exist jump deformations to all points on the “lower” spheres. In other words, there is a jump deformation from  $d_k(\lambda : \mu)$  to  $d_n(\alpha : \beta)$  for all  $n < k$ .

For the sake of simplicity,  $d_k^*$  is omitted from the diagram, although it could be viewed as an extra point at level  $k$ . From  $d_k^*$ , there exist jump deformations to  $d_{k,k}^\sharp$ , to  $d_m^*$  (for all  $1 \leq m \leq k$ ), and to  $d_{m,n}^\sharp$  (for all  $1 \leq m < n < k$ ). Furthermore, there are also jump deformations to  $d_k^*$  from each  $d_{m,n}^\sharp$  for  $k \leq m$  which are not shown here.

**5.10. Comparison to  $\mathbb{Z}_2$ -graded codifferentials.** Given the complexity of the description of the moduli space for the  $\mathbb{Z}$ -graded codifferentials, it would not be surprising if the  $\mathbb{Z}_2$ -graded picture were different. However, as we shall now show, the deformation picture in the  $\mathbb{Z}_2$ -graded case is identical to the  $\mathbb{Z}$ -graded picture.

If we consider more general codifferentials, then we have to take into account the possibility that there are terms coming from  $C_{-1}^k$  in a degree  $k$  codifferential. Thus we can express a general odd cochain in

the form

$$d = \psi_1^{k-1,1,0}a_1 + \psi_3^{k,0,0}b_1 + \psi_3^{k-2,1,1}c_1 + \psi_1^{k-1,0,1}a_2 + \psi_2^{k,0,0}b_2 + \psi_3^{k-2,1,1}c_2.$$

It is easy to check that the condition  $[d, d] = 0$  gives, among other relations, the requirements that  $b_2a_1 + a_2b_1 = 0$  and  $c_2a_1 + a_2c_1 = 0$ . If we consider a linear automorphism given by the matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & r & s \\ 0 & t & u \end{bmatrix}$ , then  $g^*(d)$  is of the form

$$g^*(d) = \psi_1^{k-1,1,0}a + \psi_3^{k,0,0}b + \psi_3^{k-2,1,1}c$$

precisely when

$$-sb_1 + ub_2 = 0, \quad sa_1 + ua_2 = 0, \quad -sc_1 + uc_2 = 0.$$

Such an automorphism always exists, so that any codifferential of degree  $k$  is equivalent to a codifferential lying in  $C_1^k$ . In fact, if we consider the standard basis  $\{e_1^k, e_1^{k-1}e_2, e_1^{k-1}e_3, e_1^{k-2}e_2e_3\}$  of  $S^k(W)$ , then

the matrix of  $d$  with respect to this basis is  $A = \begin{bmatrix} 0 & a_1 & a_2 & 0 \\ b_2 & 0 & 0 & c_2 \\ b_1 & 0 & 0 & c_1 \end{bmatrix}$ , while

the matrix of  $g : S^k(W) \rightarrow S^k(W)$  with respect to the standard basis

is  $Q = \begin{bmatrix} q^k & 0 & 0 & 0 \\ 0 & rq^{k-1} & sq^{k-1} & 0 \\ 0 & tq^{k-1} & uq^{k-1} & 0 \\ 0 & 0 & 0 & (ur-st)q^{k-2} \end{bmatrix}$ , so that the matrix of  $g^*(d)$  is

$$A' = G^{-1}AQ = \begin{bmatrix} 0 & q^{k-2}(a_1r+ta_2) & q^{k-2}(sa_1+a_2u) & 0 \\ \frac{(ub_2-sb_1)q^k}{ur-st} & 0 & 0 & q^{k-2}(uc_2-sc_1) \\ \frac{(-tb_2+rb_1)q^k}{ur-st} & 0 & 0 & q^{k-2}(-tc_2+rc_1) \end{bmatrix}.$$

Note that this matrix represents an element in  $C_1^k$  precisely when the second row and third column vanish. Now suppose that  $d$  already lies in  $C_1^k$ . It follows that  $g^*(d)$  lies in  $C_1^k$  exactly when  $s = 0$ . Thus when studying even automorphisms preserving  $C_1^k$ , we don't get a restriction just to diagonal automorphisms, but we do have a simple characterization of the automorphisms. Moreover, the action of  $g$  on  $d$  depends only on the diagonal components, which are the degree zero components, so the action is the same as the degree zero automorphisms acting in the  $\mathbb{Z}$ -graded case. As a consequence, we obtain the same equivalence classes of  $\mathbb{Z}_2$ -graded codifferentials as  $\mathbb{Z}$ -graded codifferentials.

To understand the moduli space of  $\mathbb{Z}_2$ -graded codifferentials, we need to study the deformations. We already understand the coboundaries of degree 1 and degree zero cochains, but we need to study the action of the coboundary operators on degree  $-1$  and degree  $-2$  (the action

on the degree 2 part is trivial). Let us examine the situation for each of our codifferentials.

5.10.1. *The codifferential  $d_k(\lambda : \mu)$  and  $d_{k,l}$ .*  $D : C_{-1}^1 \rightarrow C_0^{k+l-1}$  is given by  $\begin{bmatrix} 0 & \lambda & 0 \\ \lambda(l-k)+\mu & 0 & \lambda \\ 0 & l\lambda & 0 \\ 0 & \mu & 0 \end{bmatrix}$ , while that of  $D : C_{-2}^l \rightarrow C_{-1}^{k+l-1}$  is  $\begin{bmatrix} \lambda & 0 \\ 0 & -(\lambda(l-1)+\mu) \end{bmatrix}$ . For  $l > 1$ , there is exactly one cocycle of degree -1,

$$\delta_l = \psi_1^{l-1,0,1} \lambda - \psi_2^{l-2,1,1} (\lambda((l-k) + \mu)),$$

which is a coboundary when  $l \geq k$ .

Thus there are no additional extensions arising from the new cocycles, but we do have to add some additional terms to the miniversal deformation. In fact, we can generically express the miniversal deformation in the form

$$d_k^\infty(\lambda : \mu) = d_k(\lambda : \mu) + d_m(s^m, t^m) + \delta_l r^l, \quad 1 \leq m < k, 1 < l < k.$$

It is easy to analyze that terms in the self bracket of  $d_k^\infty(\lambda : \mu)$  can only arise from the brackets of the  $d_m(s^m, t^m)$  terms with the  $\delta_l r^l$  terms. We obtain relations on the base of the miniversal deformation of the form

$$\sum_{l+m=n+1} r^l (s^m(\lambda(m-k) + \mu) - t^n \lambda) = 0, \quad 2 \leq n \leq 2k-2.$$

What is the meaning of these relations? Let us use them to determine the deformations. First, let  $l_0$  be the least  $l$  such that  $r^l \neq 0$  and  $m_0$  be the least  $m$  such that either  $s^m$  or  $t^m$  does not vanish. If  $r^l$  vanishes for all  $l$ , then we are already in the same situation as the  $\mathbb{Z}$ -graded case, so we may assume otherwise. The first relation is just  $r^{l_0}(s^{m_0}(\lambda(m_0-k) + \mu) - t^{n_0} \lambda) = 0$ . If  $l_0 < m_0$ , then by applying an automorphism to  $d_k^\infty(\lambda : \mu)$ , we transform it to one in which  $l_0 > m_0$ . If  $m_0 < l_0$ , then the first relation says that  $\delta_{l_0}$  is a  $d_{m_0}(s^{m_0}, t^{m_0})$  cocycle, thus a coboundary with respect to  $\delta_{l_0}$ , which means we can eliminate it by applying an appropriate formal automorphism. Finally, if  $l_0 = m_0$ , we have exactly the relation between the coefficients which makes it possible to transform the  $m_0$  term to one which lies entirely in  $C_1^{m_0}$ . As a consequence, even though the miniversal deformation in the  $\mathbb{Z}_2$ -graded case contains additional terms, we obtain no new deformations in the  $\mathbb{Z}_2$ -graded case.

The deformation picture is similar for the extended codifferentials  $d_{k,l}$ . We omit the detailed calculations, but note that again the  $\mathbb{Z}_2$ -graded deformations are the same as for the  $\mathbb{Z}$ -graded case.

5.10.2. *The codifferential  $d_k^*$ .*  $D : C_{-1}^l \rightarrow C_0^{k+l-1}$  is given by  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ k & 0 & 0 \end{bmatrix}$ , while the matrix of  $D : C_{-2}^l \rightarrow C_{-1}^{k+l-1}$  is  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ . It follows that the only cocycle is  $\psi_2^{l,0,0}$ , which is a coboundary when  $l \geq k$ . Thus we obtain no new extensions, do have to add terms to the miniversal deformation of  $d_k^*$ .

$$(d_k^*)^\infty = d_k^* + \psi_2^{1,0,0} r_m + \psi_3^{m,0,0} s_m + (\psi_3^{n-1,1,0} + \psi_3^{n-2,1,1} k) t_n$$

$$1 \leq l, m < k, 1 < n \leq k.$$

We have

$$\frac{1}{2}[(d_k^*)^\infty, (d_k^*)^\infty] = \varphi_3^{m+n-2,1,0} s_m t_n (m-k) + \varphi_1^{l+n-1,0,0} r_l t_n$$

$$+ \varphi_2^{l+n-2,1,0} l r_l t_n + \varphi_3^{l+n-2,0,1} k r_l t_n.$$

We already saw in studying the deformations of  $d_k^*$  in the  $\mathbb{Z}$ -graded case that either  $s_m = 0$  for all  $m$  or  $t_n = 0$  for all  $n$ . In addition, we see that either  $t_n = 0$  for all  $n$  or  $r_l = 0$  for all  $n$ . These last relations are exactly what we need to be able to transform  $(d_k^*)^\infty$ , in the case when  $r_l \neq 0$  for some  $l$ , to a codifferential lying entirely in  $C_1^\bullet$ . Thus we obtain no new deformations for  $d_k^*$  in the  $\mathbb{Z}_2$ -graded case.

5.10.3. *The codifferentials  $d_k^\sharp$  and  $d_{k,l}^\sharp$ .*  $D : C_{-1}^l \rightarrow C_0^{k+l-1}$  is given by the matrix  $\begin{bmatrix} 1 & 1 & 0 \\ l & 0 & 1 \\ 0 & l & -1 \\ k & k & 0 \end{bmatrix}$ , while the matrix of  $D : C_{-2}^l \rightarrow C_{-1}^{k+l-1}$  is  $\begin{bmatrix} 1 \\ -1 \\ -(k+l-1) \end{bmatrix}$ . It follows that the only cocycles are of the form

$$\delta_l = \psi_1^{l-1,0,1} - \psi_2^{l,0,0} - \psi_2^{l-2,1,1} l,$$

which is a coboundary when  $l \geq k$ . Thus we obtain no new extensions, but as usual, do have to add terms to the miniversal deformation of  $d_k^\sharp$ .

$$(d_k^\sharp)^\infty = d_k^\sharp + (\psi_1^{0,1,0}(k-1) + \psi_3^{1,0,0} k) r + d_m(1:k) s^m + d_n^* t^n + \delta_l u^l,$$

$$1 < m < k, 1 < n < k, 1 < l < k.$$

When the self bracket of  $(d_k^\sharp)^\infty$  is computed, in addition to the relations we saw in the  $\mathbb{Z}$ -graded case, we get relations  $\sum_{l+m=n+1} u^l s^m = 0$ , which means that either all the  $\delta_l$  or all the  $d_m(1:k)$  terms must vanish in an actual deformation. But this is just what is necessary to be able to transform a deformation with  $\delta_l$  terms into one which lies in  $C_1^\bullet$ . Moreover, no new deformations arise. The situation is similar for the codifferential  $d_{k,l}^\sharp$ .

As a consequence of these calculations, we have shown that the  $\mathbb{Z}_2$ -graded moduli space and the  $\mathbb{Z}$ -graded moduli space for a vector space

of type  $(-1, 0, 1)$  are completely identical, including the deformations which glue the moduli space together. As a result, Figure 1, which illustrates how the moduli space is glued together, applies to either the  $\mathbb{Z}_2$ -graded or  $\mathbb{Z}$ -graded moduli spaces.

## 6. CODIFFERENTIALS ON A $2|1$ -DIMENSIONAL SPACE

If  $W$  is of the form  $W = W_{2x} \oplus W_{2x+1} \oplus W_{2x+2}$ , then as a  $\mathbb{Z}$ -graded space it has type  $(2x, 2x+1, 2x+2)$ , but its dimension is always  $2|1$  as a  $\mathbb{Z}_2$ -graded space. In order to list the even basis elements first, we set  $|e_1| = 2x$ ,  $|e_2| = 2x+2$  and  $|e_3| = 2x+1$ .

$$\begin{aligned} C_{2s}^r &= \langle \varphi_1^{(r-1)(x+1)+s+1, -(r-1)x-s, 0}, \varphi_2^{(r-1)(x+1)+s, -(r-1)x-s+1, 0}, \\ &\quad \varphi_3^{(r-1)(x+1)+s, -(r-1)x-s, 1} \rangle \\ C_{2s+1}^r &= \langle \psi_1^{(r-1)(x+1)+s+1, -(r-1)x-s-1, 1}, \psi_2^{(r-1)(x+1)+s, -(r-1)x-s, 1}, \\ &\quad \psi_3^{(r-1)(x+1)+s+1, -(r-1)x-s, 0} \rangle \end{aligned}$$

In particular, and of most interest to us are the following.

$$\begin{aligned} C_0^r &= \langle \varphi_1^{r+(r-1)x, -x(r-1), 0}, \varphi_2^{(r-1)(x+1), 1-x(r-1), 0}, \varphi_3^{(r-1)(x+1), -x(r-1), 1} \rangle \\ C_1^r &= \langle \psi_1^{(r-1)(x+1)-1, -x(r-1)-1, 1}, \psi_2^{(r-1)(x+1), -x(r-1), 1}, \psi_3^{(r-1)(x+1)+1, -x(r-1), 0} \rangle \end{aligned}$$

There is a natural basis of  $S^k(W)$ , given by

$$\{e_1^k, e_1^{k-1}e_2, \dots, e_2^k, e_1^{k-1}e_3, e_1^{k-2}e_2e_3, \dots, e_2^{k-1}e_3\},$$

from which it follows that  $S^n(W)$  has  $\mathbb{Z}_2$ -graded dimension  $(k+1)|k$ . If  $A$  is the matrix of an odd codifferential of degree  $k$ , then  $A$  can be expressed in block form as  $A = \begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}$ . It can be checked that  $A$  is a codifferential precisely when either  $A_1$  or  $A_2$  vanishes. A codifferential for which  $A_2$  vanishes is said to be of the *first kind*, and one for which  $A_1$  vanishes is of the *second kind*. More generally, a codifferential with terms in more than one degree will still have only terms of the first kind or second kind (see [1]). Thus the moduli space of all codifferentials splits into disjoint subspaces, the moduli spaces of codifferentials of the first and second kinds. We will study these spaces separately.

**6.1. Codifferentials on a vector space of type  $(0,1,2)$ .** Suppose  $|e_1| = 0$ ,  $|e_2| = 2$ , and  $|e_3| = 1$ . Then  $\varphi_n^{k,l,m} = |e_n| - 2l - m$  so

$$\begin{aligned} C_0^r &= \langle \varphi_1^{r,0,0}, \varphi_2^{r-1,1,0}, \varphi_3^{r-1,0,1} \rangle \\ C_1^r &= \langle \psi_2^{r-1,0,1}, \psi_3^{r,0,0} \rangle. \end{aligned}$$

Up to equivalence there are only two codifferentials of degree  $k$ , a codifferential of the first kind given by  $d = \psi_2^{k-1,0,1}$  and a codifferential of the second kind,  $d = \psi_3^{k,0,0}$ .

**6.1.1. Codifferentials of the first kind.** Consider first the case

$$(11) \quad d_k = \psi_2^{k-1,0,1}.$$

The cohomology of this codifferential is given by

$$\begin{aligned} H_1^n &= \langle \psi_2^{n-1,0,1} \rangle, & n < k \\ H_1^n &= 0, & \text{otherwise.} \end{aligned}$$

Since  $H_1^n$  vanishes for  $n \geq k$  it follows that there are no nontrivial extensions of  $d$ . Moreover

$$d_k^\infty = \psi_2^{k-1,0,1} + \sum_{m=1}^{k-1} \psi_2^{m-1,0,1} t_m,$$

so this means that the codifferentials of the first kind form a discrete family with codifferentials of degree  $k$  deforming into codifferentials of degree  $m$  for  $m < k$  by jump deformations.

**6.1.2. Codifferentials of the second kind.** For the codifferential of the second kind of degree  $k$ ,

$$(12) \quad d_k = \psi_3^{k,0,0},$$

the pattern is similar. In this instance we have

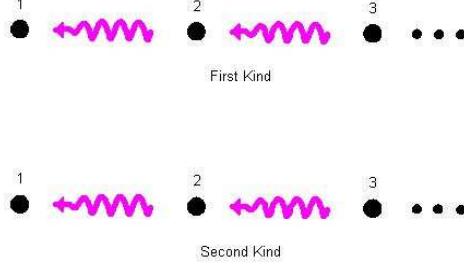
$$\begin{aligned} H_1^n &= \langle \psi_3^{n,0,0} \rangle, & n < k \\ H_1^n &= 0, & \text{otherwise} \end{aligned}$$

and

$$d_k^\infty = \psi_3^{k,0,0} + \sum_{m=1}^{k-1} \psi_3^{m,0,0} t_m.$$

Thus, the codifferentials of the second kind form another discrete family. Figure 2 is an illustration of the moduli space of codifferentials.

Since the space of  $\mathbb{Z}_2$ -graded codifferentials is the same for this example and for the example in the next section, we will defer the analysis of the moduli space of  $\mathbb{Z}_2$ -graded codifferentials until then. However, it is important to note that for the  $\mathbb{Z}$ -graded moduli space of type (0,1,2) considered here, the map to the  $\mathbb{Z}_2$ -graded moduli space is injective, but not surjective.



**FIGURE 2. The moduli space of codifferentials on  $\mathbf{W}_0 \oplus \mathbf{W}_1 \oplus \mathbf{W}_2$ .** This moduli space contains two discrete families of codifferentials. The upper diagram shows the codifferentials of the first kind, in which each  $d_k$  has a jump deformation to each “lower”  $d_m$  (for all  $m < k$ ). The lower diagram represents the codifferentials of the second kind, which follow an analogous pattern.

**6.2. Codifferentials on a vector space of type  $(-2, -1, 0)$ .** Suppose  $|e_1| = -2$ ,  $|e_2| = 0$  and  $|e_3| = -1$ . Then

$$\begin{aligned} C_0^r &= \langle \varphi_1^{1,r-1,0}, \varphi_2^{0,r,0}, \varphi_3^{0,r-1,1} \rangle \\ C_1^r &= \langle \psi_1^{1,r-2,1}, \psi_2^{0,r-1,1}, \psi_3^{1,r-1,0} \rangle, \end{aligned}$$

so we have a more complicated situation among the codifferentials of the first kind, which are of the form

$$(13) \quad d_k(\lambda : \mu) = \psi_1^{1,k-2,1} \lambda + \psi_2^{0,k-1,1} \mu,$$

where, for  $k = 1$  we omit the first term ( $\lambda = 0$ ). Using the diagonal linear automorphism  $g = \text{diag}(p, q, r)$ , it is easy to see that

$$g^*(d_k(\lambda : \mu)) = q^{k-2} r d_k(\lambda : \mu),$$

so the only equivalent  $\mathbb{Z}$ -graded codifferentials are multiples. The situation is quite different if we consider the even automorphism  $g$  of  $W$  which interchanges  $e_1$  and  $e_2$ . This automorphism is even, but not of

degree 0 and

$$g^*(d_2(\lambda : \mu)) = d_2(\mu : \lambda).$$

This means that  $d_2(\lambda : \mu) \sim d_2(\mu : \lambda)$  as  $\mathbb{Z}_2$ -graded codifferentials, but not as  $\mathbb{Z}$ -graded codifferentials. This example shows that the map from the moduli space of  $\mathbb{Z}$ -graded codifferentials to the moduli space of  $\mathbb{Z}_2$ -graded codifferentials is not injective. Note that the map from the  $\mathbb{Z}$ -graded moduli space in the previous section is injective, because the  $\mathbb{Z}$ -graded space in that case does not include codifferentials on which the map fails to be injective.

The  $\mathbb{Z}$ -graded codifferentials of the first kind are labeled projectively by points  $(\lambda : \mu) \in \mathbb{P}^1(\mathbb{C})$ . The matrix of  $D : C_0^l \rightarrow C_1^{k+l-1}$  is

$$\begin{bmatrix} \lambda & (k-2)\lambda & \mu(l-1) \\ \mu & \mu(k-l-1) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank of this matrix is equal to 2 except when  $l = 1$  or  $\mu = 0$ , when the rank is 1. This means, that the codifferential  $d_k(1 : 0)$  is special. We will study the extensions of the generic case first.

**6.3. Extensions of  $d_k(\lambda : \mu)$ ,  $\mu \neq 0$ .** We have

$$\begin{aligned} H_1^1 &= \langle \psi_2^{0,0,1} \rangle \\ H_1^n &= \langle \psi_1^{1,n-2,1}, \psi_2^{0,n-1,1} \rangle, \quad 1 < n < k \\ H_1^k &= \langle \psi_2^{0,k-1,1} \rangle \\ H_1^n &= 0, \quad n > k. \end{aligned}$$

Because of this, there are no nontrivial extensions of  $d_k(\lambda : \mu)$ . Note that when  $k = 1$ , we must have  $\lambda = 0$ , and then  $H_1^n = 0$  for all  $n \geq 1$ , so that  $d_1(0 : 1)$  has no extensions or deformations.

**6.4. Extensions of  $d_k(1 : 0)$ .** We have

$$\begin{aligned} H_1^1 &= \langle \psi_2^{0,0,1} \rangle \\ H_1^n &= \langle \psi_1^{1,n-2,1}, \psi_2^{0,n-1,1} \rangle, \quad 1 < n < k \\ H_1^n &= \langle \psi_2^{0,n-1,1} \rangle, \quad n \geq k. \end{aligned}$$

Because the cohomology does not vanish identically for  $n > k$ , there are nontrivial extensions of  $d(1 : 0)$ . If  $l > k$ , the extension

$$d_{k,l} = \psi_1^{1,k-2,1} + \psi_2^{1,l-1,1}x$$

is nontrivial when  $x \neq 0$ . Note that if  $g = \text{diag}(1, q, q^{2-k})$ , then

$$g^*(d_{k,e}) = \psi_1^{1,k-2,1} + \psi_2^{1,l-1,1}q^{l-k}x,$$

so that when  $x \neq 0$  we can assume that  $x = 1$ . Moreover, one can show that  $H_1^n(d_{k,l}) = 0$  for  $n > l$ , so that  $d_{k,l}$  itself has no nontrivial extensions.

**6.4.1. Deformations of codifferentials of the first kind.** A codifferential of the first kind of degree  $k$  is equivalent to  $d_k(\lambda : \mu)$  or  $d_{k,l}$ . We will study the deformations of this space in order to obtain a more detailed picture of the moduli space of codifferentials of the first kind. If  $k > 1$ , then  $H_1^k = \langle \psi_2^{0,k-1,1} \rangle$ , so we always have a deformation along the codifferentials of degree  $k$ ; in other words, we can deform along the family. If  $\mu \neq 0$ , this completes the picture in terms of deformations to codifferentials of the same order. However, for  $d_k(0 : 1)$ , we can deform to  $d_{k,l}$  for any  $l > k$ . On the other hand,  $d_{k,l}$  deforms to  $d_{k,m}$  for every  $k < m < l$ , because  $\psi_1^{1,m-2,1}$  remains a cohomology class for  $d_{k,l}$ .

We can also deform to lower order codifferentials. The codifferential  $d_k(\lambda : \mu)$  deforms to  $d_m(\alpha : \beta)$  for any  $(\alpha : \beta)$  and all  $1 < m < k$ , as well as to  $d_{m,n}$  if  $1 < m < n \leq k$ . Similarly,  $d_{k,l}$  deforms to  $d_m(\alpha : \beta)$  for any  $(\alpha : \beta)$  and all  $1 < m < k$ , as well as to  $d_{m,n}$  if  $1 < m \leq l$ . Finally, everything deforms to  $d_1(0 : 1)$ , which has no deformations.

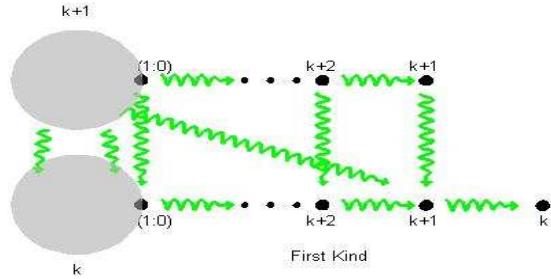
**6.4.2. Deformations of Codifferentials of the second kind.** A codifferential of the second kind is equivalent to one with leading term of the form  $d_k = \psi_3^{1,k-1,0}$ . Since

$$\begin{aligned} H^n(d_k)_1 &= \langle \psi_3^{1,n-1,0} \rangle, & n < k \\ H^n(d_k)_1 &= 0, & n \geq k, \end{aligned}$$

it follows that  $d_k$  deforms to  $d_n$  precisely when  $n < k$ . Thus the moduli space of codifferentials of the second kind is another discrete chain, with jump deformations to lower elements on the chain. A picture of the moduli space appears in figure 3.

## 7. CONCLUSION

Applications of infinity algebras occur in both  $\mathbb{Z}$ -grading and  $\mathbb{Z}_2$ -grading, so it is important to understand the relationship between them. We showed in this paper that even for 3-dimensional infinity algebras this relationship is not trivial. Sometimes (as seen in section 5) the natural map between the moduli space of  $\mathbb{Z}$ -graded infinity algebras and the moduli space of  $\mathbb{Z}_2$ -graded infinity algebras is an isomorphism. In the example in section 5, the moduli spaces are quite complicated, but some seemingly unlikely coincidences conspire to yield identical



**FIGURE 3. The moduli space of codifferentials on  $\mathbf{W}_{-2} \oplus \mathbf{W}_{-1} \oplus \mathbf{W}_0$ .** The bottom row, depicting level  $k$ , contains a sphere  $d_k(\lambda : \mu)$  with marked point  $d_k(1 : 0)$ , an infinite line of dots  $\{d_{k,m} | k \leq m\}$ , and horizontal arrows which show that  $d_{k,l}$  deforms to  $d_{k,m}$  for  $k < m < l$ . Although only one other row (depicting level  $k+1$ ) is shown, similar rows exist for all  $k \in \mathbb{N}$ . Each arrow indicates the direction of a jump deformation. For example, the arrows between the large spheres signify that  $d_{k+1}(\lambda : \mu)$  can deform to  $d_k(\alpha : \beta)$ , for all values of  $\alpha$  and  $\beta$ . and the remaining downward and diagonal arrows represent that  $d_{k,l}$  deforms to  $d_{m,n}$  if  $1 < m < l$ . Although  $d_{k,l}$  deforms to  $d_m(\alpha : \beta)$  for all  $1 < m < k$ , the arrows are omitted in order to simplify the diagram. Similarly, although  $d_k(\lambda : \mu)$  can deform to  $d_m(\alpha : \beta)$  for all  $1 < m < k$ , an infinite number of these arrows between the spheres are omitted as well.

moduli spaces. In fact, there are twice as many  $\mathbb{Z}_2$ -graded codifferentials, but the group of equivalences is just larger enough to identify them with the  $\mathbb{Z}$ -graded codifferentials.

In general, the space of  $\mathbb{Z}$ -graded codifferentials will be much sparser than the  $\mathbb{Z}_2$ -graded codifferentials, as is illustrated by the example in section 6.1. In that example, the map from the moduli space of  $\mathbb{Z}$ -graded codifferentials to the space of  $\mathbb{Z}_2$ -graded codifferentials was at

least injective. When the moduli space of  $\mathbb{Z}$ -graded codifferentials is sparse, it is not really surprising that the map should be injective, but not surjective.

In the example in section 6.2, the map from the moduli space of  $\mathbb{Z}$ -graded codifferentials fails to be injective in a very interesting manner. The map takes a stratum of the  $\mathbb{Z}$ -graded moduli space which is a  $\mathbb{P}^1$  to an orbifold stratum of the  $\mathbb{Z}_2$ -graded moduli space given by a  $\mathbb{P}^1/\Sigma_2$ , due to an extra symmetry induced by the larger group of equivalences in the  $\mathbb{Z}_2$ -graded case. Thus even though the map fails to be injective, the  $\mathbb{Z}_2$ -graded moduli space is closely related to the  $\mathbb{Z}$ -graded moduli space.

Note that in each 3-dimensional  $L_\infty$  algebra examined here, we have shown that the moduli space is decomposable in a unique manner as a union of orbifolds (cf. [7]), with the distinct strata glued together by jump deformations. In each example, the map between the moduli spaces respects this stratification, and in the one example where it fails to be injective, the map has a very interesting structure on the stratum on which it fails to be injective.

From the examples, we can deduce that the study of moduli spaces of  $\mathbb{Z}$ -graded and  $\mathbb{Z}_2$ -graded moduli spaces cannot be reduced to a study of only one of these types. Nevertheless, it may be possible to deduce important information about the  $\mathbb{Z}$ -graded moduli space from that of the  $\mathbb{Z}_2$ -graded one. Explorations of higher dimensional examples may reveal more information about this connection.

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